FACTORIZATION OF SPARSE POLYNOMIALS

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Sparse Polynomials & Factorization

$f = \gamma_0 + \gamma_1 t^{a_1} + \cdots + \gamma_N t^{a_N} \in K[t^{\pm 1}]$ sparse

Let $g \mid f$. Is g sparse too?

<u>Not</u> necessarily:

$$1 - x^n = (1-x) \cdot (1 + x + x^2 + \cdots + x^{n-1})$$

Reducibility of Binomials

I (Vahlen 1895, Capelli 1898, Rédei 1967)

TFAE

- $\gamma - x^n$ reducible
- either

$\exists\, p \mid n$ and $\beta \in K$ st $\gamma = \beta^p$

$$\gamma - x^n = (\beta - x^p)(\beta^{p-1} + \beta^{p-2} x^p + \dots + x^{n-p})$$

or

$4 \mid n$ and $\exists\, \beta \in K$ st $\gamma = -4\beta^4$

$$\gamma - x^n = (2\beta^2 - 2\beta x^{n/4} + x^{n/2}) \cdot (2\beta^2 + 2\beta x^{n/4} + x^{n/2})$$

Sparse Polynomials in Families

Fix $\underline{\gamma} \in K^{N+1}$ and vary $\underline{a} \in \mathbb{Z}^N$

Set $f_{\underline{a}} = \gamma_0 + \gamma_1 t^{a_1} + \dots + \gamma_N t^{a_N}$

i.e. $f_{\underline{a}} = L(t^{a_1}, \dots, t^{a_N})$ with $L = \gamma_0 + \gamma_1 x_1 + \dots + \gamma_N x_N$

$$
\begin{tikzcd}
\mathbb{C}^\times \arrow[r, "t \mapsto (t^{a_1}, \dots, t^{a_N}) \text{ varying}"'] \arrow[rr, bend right, "f_{\underline{a}}"'] & (\mathbb{C}^\times)^N \arrow[r, "L \text{ fixed}"] & \mathbb{C}
\end{tikzcd}
$$

Families of Factors

Let $M \in \mathbb{Z}^{N\times N}$ st $L(\underline{x}^M) = P(\underline{x}) \cdot Q(\underline{x})$

If $\exists\, \underline{b} \in \mathbb{Z}^N$ st $\underline{a} = M \cdot \underline{b}$

$$\Longrightarrow \quad f_{\underline{a}} = L(t^{\underline{a}}) = L(t^{M\cdot\underline{b}}) = P(t^{\underline{b}}) \cdot Q(t^{\underline{b}})$$

<u>Ex.:</u> Set $L = 4 + x$

We have $4 + x^4 = (2 + 2x + x^2) \cdot (2 - 2x + x^2)$

Let $a \in \mathbb{Z}$ st $a = 4 \cdot b$

$$\Longrightarrow \quad f_a = 4 + t^a = (2 + 2t^b + t^{2b}) \cdot (2 - 2t^b + t^{2b})$$

Families of Factors (cont.)

Set $K = \mathbb{Q}$

<u>Conj.</u> (Schinzel 1965)

Let $L = \gamma_0 + \gamma_1 x_1 + \cdots + \gamma_N x_N$ as before

$\exists\ \Omega \subset \mathbb{Z}^{N\times N}$ finite st

$\forall\ \underline{a} \in \mathbb{Z}^N\ \exists\ M \in \Omega,\ \underline{b} \in \mathbb{Z}^N$ st $\underline{a} = M\cdot\underline{b}$

and if $L(\underline{x}^M) = \prod_p p^{e_p}$ irred factorization

$$\Longrightarrow \frac{f_{\underline{a}}}{\mathrm{cyclo}(f_{\underline{a}})} = \prod_p \frac{P(t^{\underline{b}})^{e_p}}{\mathrm{cyclo}(P(t^{\underline{b}}))} \text{ irred factorization}$$

mxl cyclotomic factor

OK for $N=1$ (Schinzel 1965) and $N=2$ (Filaseta 1999)

A Function Field Analogue

$K = \mathbb{C}(s)$

T (Amoroso - S.)

Let $L = \gamma_0 + \gamma_1 x_1 + \cdots + \gamma_N x_N \in \mathbb{C}(s)[\underline{x}^{\pm 1}]$

$\exists\ \Omega \subset \mathbb{Z}^{N \times N}$ finite st

$\forall\ \underline{a} \in \mathbb{Z}^N\ \exists\ M \in \Omega,\ \underline{b} \in \mathbb{Z}^N$ st $\underline{a} = M \cdot \underline{b}$

and if $L(\underline{x}^M) = \prod_P P^{e_P}$ irred factorization

$$\Rightarrow \frac{f_{\underline{a}}}{\mathrm{const}(f_{\underline{a}})} = \prod_P \frac{P(t^{\underline{b}})^{e_P}}{\mathrm{const}\,(P(t^{\underline{b}}))} \quad \text{irred factorization}$$

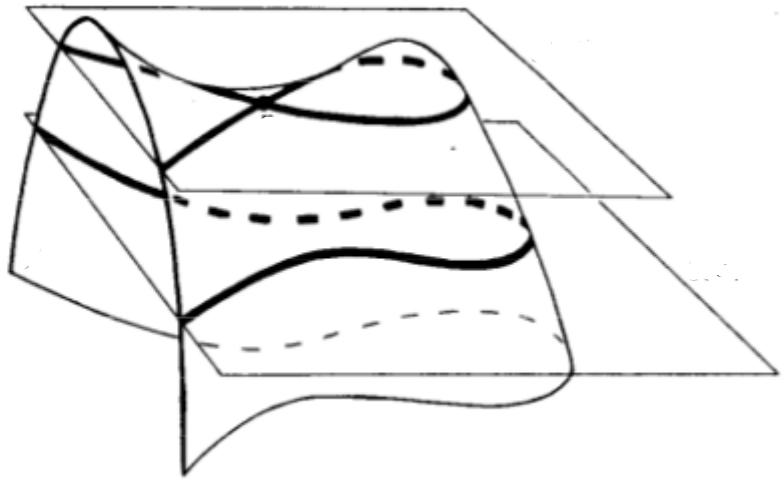
mxl factor in $\mathbb{C}[t^{\pm 1}]$

Alternatively:

$$\prod_p p(t^{b})^{e_p}$$

is the irreducible factorization of f_2 in $\mathbb{C}(s,t)^\times / \mathbb{C}(t)^\times$

Bertini's Theorem



from Hartshorne: AG, p 179

$$\begin{array}{l} Y \text{ irreducible} \\ \downarrow \pi \\ \mathbb{C}^n \end{array}$$

$E \subset \mathbb{C}^n$ generic with $\operatorname{codim} E < \dim \pi Y$

$\Rightarrow \pi^{-1} E$ irreducible

Let $\begin{array}{c} Y \\ \big\downarrow \\ G \end{array}$ irreducible

algebraic group

$E \subset G$ "generic" coset of codim $< \dim \pi(Y)$

Is $\pi^{-1}E$ irreducible ? <u>Not necessarily...</u>

<u>Ex.</u>: $F = s^2 - x_1 x_2^2 \in \mathbb{C}[s^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}]$

$$\begin{array}{ccc} Y = (F=0) \subset (\mathbb{C}^\times)^3 & & (s, x_1, x_2) \\ \big\downarrow & & \big\downarrow \\ (\mathbb{C}^\times)^2 & & (x_1, x_2) \end{array}$$

$(a_1, a_2) \in \mathbb{Z}^2 \Rightarrow \pi^{-1}\left(t \mapsto (t^{a_1}, t^{a_2})\right) = \left(F(s, t^{a_1}, t^{a_2}) = 0\right)$

reducible for a_1 even

A Toric Bertini Theorem for Covers

I Let $\begin{array}{c} Y \text{ irreducible} \\ \downarrow \pi \text{ dominant} \\ (\mathbb{C}^\times)^n \end{array}$

$\Rightarrow \exists\ \mathcal{F} \subset (\mathbb{C}^\times)^n$ finite union of proper subgroups of $(\mathbb{C}^\times)^n$

Ω finite collection of isogenies $(\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^n$

s.t. $\forall\ E = p \cdot T \subset (\mathbb{C}^\times)^n$ proper coset

one of the following holds:

(1) $T \subset \mathcal{F}$

(2) $\exists\ \varphi \in \Omega$ st T factors through φ and $\varphi^* Y$ is reducible

(3) $\pi^{-1} E$ irreducible

Proof: based on previous results by Zannier (2011)

and Fuchs, Mantova and Zannier (2017)

A Possible Generalization (work in progress)

Let $\delta_0 + \delta_1 y_1 + \cdots + \delta_M y_M \in \mathbb{C}[\underline{y}]$ fixed

For $\underline{a}, \underline{b} \in \mathbb{Z}^M$

set $f_{\underline{a},\underline{b}} = \delta_0 + \delta_1 s^{a_1} t^{b_1} + \cdots + \delta_M s^{a_M} t^{b_M} \in \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$

The "true" bivariate irreducible factors of the $f_{\underline{a},\underline{b}}$'s should be given by a finite number of families

Would follow from a toric Bertini theorem for a map

$$Y \to (\mathbb{C}^\times)^n$$

not necessarily dominant

Thanks!